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AdS Geometry, Projective Embedded Coordinates and Associated Isometry Groups

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This work is intended to investigate the geometry of anti-de Sitter spacetime (AdS), from the point of view of the Laplacian Comparison Theorem (LCT), and to give another description of the hyperbolical embedding standard formalism of the de Sitter and anti-de Sitter spacetimes in a pseudoeuclidean spacetime. After Witten proved that general relativity is a renormalizable quantum system in $(1 + 2)$ dimensions, it is possible to point out few interesting motivations to investigate AdS spacetime. A lot of attempts were made to generalize the gauge theory of gravity in $(1 + 2)$ dimensions to higher ones. The first one was to enlarge the Poincaré group of symmetries, supposing an AdS group symmetry, which contains the Poincaré group. Also, the AdS/CFT correspondence asserts that a maximal supersymmetric Yang–Mills theory in four-dimensional Minkowski spacetime is equivalent to a type IIB closed superstring theory. The 10-dimensional arena for the type IIB superstring theory is described by the product manifold $S^5 \times$ AdS, an impressive consequence that motivates the investigations about the AdS spacetime in this paper, together with the de Sitter spacetime. Classical results in this mathematical formulation are reviewed in a more general setting together with the isometry group associated to the de Sitter spacetime. It is known that, out of the Friedmann models that describe our universe, the Minkowski, de Sitter, and anti-de Sitter spacetimes are the unique maximally isotropic ones, so they admit a maximal number of conservation laws and also a maximal number of Killing vectors. In this paper it is shown how to reproduce some geometrical properties of AdS, from the LCT in AdS, choosing suitable functions that satisfy basic properties of Riemannian geometry. We also introduce and discuss the well-known embedding of a four-sphere and a four-hyperboloid in a five-dimensional pseudoeuclidean spacetime, reviewing the usual formalism of spherical embedding and the way how it can retrieve the Robertson–Walker metric. With the choice of the de Sitter metric static frame, we write the so-called reduced model in suitable coordinates. We assume the existence of projective coordinates, since de Sitter spacetime is orientable. From these coordinates, obtained when stereographic projection of the de Sitter four-hemisphere is done, we consider the Beltrami geodesic representation, which gives a more general formulation of the seminal full model described by Schrödinger, concerning the geometry and the topology of de Sitter spacetime. Our formalism retrieves the classical one if we consider the

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metric terms over the de Sitter splitting on Minkowski spacetime. From the covariant derivatives we find the acceleration of moving particles, Killing vectors and the isometry group generators associated to de the Sitter spacetime.

KEY WORDS: Laplacian comparison theorem; de Sitter and anti-de Sitter spacetimes; Beltrami representation; Killing vectors.

1. INTRODUCTION

Today it is of great interest to investigate isometry groups of a given universe model, and it becomes natural to ask whether some models admit the energy conservation law. We restrict our attention to de Sitter (dS) and anti-de Sitter (AdS) spacetimes, which are respectively solutions of Einstein equations with cosmological constant $\Lambda = \pm 3/R^2$ ($R > 0$), and curvature given by the components $R_{\mu\nu} = \Lambda g_{\mu\nu}$ of the Ricci tensor. These manifolds have shown themselves suitable as geometric arenas to investigate conformal field theories (Di Francesco *et al.*, 1996) and superstrings (Kaku, 2000). Since de Sitter group is the maximal inner group contained in the conformal group, many physical theories are formulated in dS and AdS scenarios. The maximally compact subgroup of SO(3,2), the symmetry group of the $\mathrm{dS}_{3,2}$ spacetime, is SO(3)×SO(2), which is twofold covered by SU(2)×U(1). It can therefore be used as an alternative formalism to describe the Glashow–Weinberg–Salam model of electroweak interactions (Weinberg, 1967), since the gauge group SU(2)×U(1) is related to the isospin and weak hypercharge of elementary particles. The group SO(3,2) is also a dynamical group associated to the *Zitterbewegung* (Barut and Bracken, 1981; Huang, 1992). dS and AdS spacetimes also allow exact solutions of the field equations and the symmetry group SO(4,1) is used to classify physical states (Börner and Dürr, 1969; Prasad, 1966).

dS spacetime is geometrically described as a four-hyperboloid with topology $S^3 \times \mathbb{R}$. If a Wick rotation of the time coordinate is performed, dS can be seen as a four-sphere, and then it is possible, using the methods of projective geometry (Arcidiacono, 2000), to describe dS spacetime in terms of Minkowski orthogonal coordinates. From now on $\mathrm{dS}_{4,1}$ denotes the de Sitter spacetime embedded in a pseudoeuclidean spacetime endowed with a metric of signature (4,1).

This article is organized as follows: In Section 2 the metric in AdS is obtained, describing the geometry of a simply connected hyperbolic spacetime of constant sectional curvature k. The Laplacian Comparison Theorem is also proved, from the viewpoint of AdS geometry, which is briefly reviewed. In Section 3 we present and discuss the main differences between the spherical and hyperbolical embedding in a pseudoeuclidean spacetime, retrieving the Robertson–Walker metric by the introduction of an appropriate expansion parameter. Some of the features of the Schrödinger model (Schrödinger, 1957) are reviewed, together with a brief exposition of the de Sitter metric, using the static frame. In Section 4, after the metric of $dS_{4,1}$ is constructed, the covariant derivative and subsequently the acceleration of a moving particle are explicitly given in terms of the Christoffel connection symbols. The Killing vectors are obtained from the Killing equations, and also the generators of the isometry group of $dS_{4,1}$. In Section 5, after performing a Wick rotation in time coordinate, $dS_{4,1}$ spacetime is described as a four-sphere and the Beltrami representation is obtained. It gives an useful way to describe embedded projective coordinates as the usual orthogonal coordinates on Minkowski spacetime. The de Sitter metric, which is retrieved if we consider the linear metric tensor components on Minkowski spacetime, is generalized.

2. REVISITING THE Ads GEOMETRY AND THE LAPLACIAN COMPARISON THEOREM

In this section we revisit the approach given in Escobar (2000) in the case when the manifold M is given by the hyperbolic AdS spacetime. From now on we denote $\partial_\alpha = \partial/\partial_\alpha$ and we call by *metric*, a nondegenerate symmetric bilinear form.

Consider a C^2 function $\psi : M \to \mathbb{R}$ and vector fields $X, Y \in \Omega(M)$. The hessian of ψ is defined as

$$H\psi(X, Y) = D_X\, d\psi(Y) = XY(\psi) - D_X Y(\psi). \tag{1}$$

The metric in $\mathbb{R}^n$ is given by

$$g = dr \otimes dr + \psi^2(r)\, d\Omega \otimes d\Omega, \tag{2}$$

where $d\Omega \otimes d\Omega : T_x(S^{n-1}) \times T_x(S^{n-1}) \to \mathbb{R}$ is the metric on the $(n-1)$-sphere S^{n-1}, for $x = r\Omega$, $r > 0$, $\Omega \in S^{n-1}$. The tangent space at $x \in S^{n-1}$ is denoted by $T_x(S^{n-1})$. From the definition of the hessian, and using the fact that the curves $r\Omega$ for Ω fixed are geodesics, and ∂_r is the velocity vector, we have

$$Hr\,(\partial_r, \partial_r) = \partial_r \partial_r r - D_{\partial_r} \partial_r r = 0. \tag{3}$$

Besides, if $X \cdot \partial_r = 0$, the expression

$$Hr\,(\partial_r, X) = Hr\,(X, \partial_r) = X\,(\partial_r r) - (D_X \partial_r)\, r \tag{4}$$

follows. But since $(D_X \partial_r) r = (D_X \partial_r) \cdot \partial_r$, if we suppose that the radial geodesics are parametrized by the arc lenght, i.e.,

$$\partial_r \cdot \partial_r = 1, \tag{5}$$

we can covariantly differentiate Eq. (5) with respect to X, obtaining $(D_X \partial_r) \cdot \partial_r = 0$. Therefore,

$$Hr\,(\partial_r, X) = 0. \tag{6}$$

Let $a \in \mathbb{R}^*_+$, and suppose that the vector fields X and Y are tangent to the level hypersurface $r = a$. Then

$$Hr(X, Y) = -(D_X Y)(r) = -(D_X Y) \cdot \partial_r = Y \cdot (D_X \partial_r), \tag{7}$$

since $Y \cdot \partial_r = 0$. Now, let $\{e_i\}_{i=1}^{n-1}$ be a set of orthonormal (coordinate) frame on S^{n-1}, and without loss of generality, let $e_n := \partial_r$ be the normal vector field to the hypersurface $r = c$. On one hand we have

$$e_i \cdot (D_{e_j} \partial_r) = e_i \cdot (D_{e_j} e_n) = \Gamma^p_{jn} e_p \cdot e_i = \Gamma_{jin}, \tag{8}$$

where Γ_{abc} are the Christoffel symbols. On the other hand, these symbols are also given by

$$\begin{aligned}\Gamma_{jin} &= \frac{1}{2}(\partial_r g_{ij} + \partial_{x^j} g_{in} - \partial_{x^i} g_{jn}) = \frac{1}{2}\partial_r(\psi^2 \mathring{g}_{ij}) = \psi\psi' \mathring{g}_{ij} = \frac{\psi'}{\psi}(\psi^2 \mathring{g}_{ij}) \\ &= \frac{\psi'}{\psi} g_{ij}\end{aligned} \tag{9}$$

where $\mathring{g} := d\Omega \otimes d\Omega$ denotes the metric in S^{n-1}. So the following proposition has just been proved:

Proposition 1 *Let* $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, *given explicitly by* $g = dr \otimes dr + \psi^2(r)\, d\Omega \otimes d\Omega$, *be the metric in* $\mathbb{R}^n$, *where* $d\Omega \otimes d\Omega : T_x(S^{n-1}) \times T_x(S^{n-1}) \to \mathbb{R}$ *denotes the metric in* S^{n-1}. *Consider the distance function* $r = r(x)$, *then for* $x = r\Omega, r > 0$, *and* $\Omega \in S^{n-1}$ *the relation*

$$Hr(x) = \frac{\psi'(r)}{\psi(r)}(g - dr \otimes dr) \tag{10}$$

is verified, and the laplacian of r is given by

$$\Delta r(x) = (n-1)\frac{\psi'(r)}{\psi(r)}. \tag{11}$$

Suppose now that the metric in $\mathbb{R}^n$ is given by

$$g = dr \otimes dr + \psi_k^2(r)\, d\Omega \otimes d\Omega, \tag{12}$$

where

$$\psi_k(r) = \begin{cases} \frac{1}{\sqrt{-k}} \sin h(\sqrt{-k}r) & k < 0, \\ r & k = 0, \\ \frac{1}{\sqrt{k}} \sin(\sqrt{k}r) & k > 0. \end{cases} \tag{13}$$

In the first case ($k < 0$) a metric in AdS is obtained, and AdS does correspond to a simply connected hyperbolic spacetime of constant sectional curvature k. It is immediate that

$$\frac{d\psi_k(r)}{dr} = \begin{cases} \cosh(\sqrt{-k}r) & k < 0, \\ 1 & k = 0, \\ \cos(\sqrt{k}r) & k > 0. \end{cases} \tag{14}$$

From now on we are concerned about the case $k < 0$, describing the AdS geometry. Let r_k the geodesic distance to the origin with respect to the metric (12). From Eq. (14), we obtain

$$H\alpha_k = \sqrt{-k}\ \coth(\sqrt{-k}r)(g - dr \otimes dr), \tag{15}$$

and then it follows that

$$\frac{1}{n-1}\Delta r_k = \begin{cases} \frac{1}{\sqrt{-k}}\coth(\sqrt{-k}r) & k < 0, \\ 1/r & k = 0, \\ \frac{1}{\sqrt{k}}\cot(\sqrt{k}r) & k > 0. \end{cases} \tag{16}$$

This result shall still be used. Now we assert and show some properties of the AdS geometry, based on the Laplacian Comparison Theorem (for more details, see Escobar, 2000).

Theorem 1. *Let X, Y be vector fields in the manifold AdS. Assume that the Ricci curvature in all points of the AdS manifold satisfies $\mathrm{Ric}(X, Y) \geq (n-1)gk$. If r denotes the geodesic distance to a point $p \in AdS$ and if it is a differentiable function at $x \in AdS$, then*

$$\Delta r(x) \leq \Delta r_k(s), \tag{17}$$

where $s = r(x)$. Equality holds only if any sectional curvature along the minimizing geodesic between p and x, that contains r, is constant and equal to k.

Proof: Let $\alpha \in C^3(\mathrm{AdS})$ and $\{e_i\}_{i=1}^n$ be an orthonormal frame field in a tangent space at a point $p \in$ AdS. Then $||\nabla\alpha||^2 = f_i f^i$. For each $j = 1, \ldots, n$ it can be verified that

$$\left(\frac{1}{2}||\nabla\alpha||^2\right)_j = (\alpha^i\alpha_i)_j. \tag{18}$$

Hence, if Δ denotes the laplacian, it follows that

$$\frac{1}{2}\Delta(||\nabla\alpha||^2) = \left(\frac{1}{2}\sum_{j=1}^n ||\nabla\alpha||^2\right)_{jj}$$
$$= \sum_{i,j=1}^n (\alpha_{ij}\alpha_{ij} + \alpha_i\alpha_{ijj}). \tag{19}$$

Since the Ricci equation asserts that $\alpha_{ikl} = \alpha_{ilk} + R^s_{kli} f_s$ (Ornea and Ormani, 1995), where R_{klis} denotes the coefficients of the Riemann curvature tensor, it follows that

$$\frac{1}{2}\Delta(||\nabla\alpha||^2) = \sum_{i,j=1}^{n} \left(\alpha_{ij}^2 + \alpha_i\alpha_{jji} + R_{ij}\alpha_i\alpha_j\right), \tag{20}$$

which implies that

$$\frac{1}{2}\Delta(||\nabla\alpha||^2) = ||H\alpha||^2 + \nabla(\Delta\alpha)\cdot\nabla\alpha + \mathrm{Ric}(\nabla\alpha, \nabla\alpha). \tag{21}$$

In the particular case when $||\nabla\alpha|| = 1$, then $||H\alpha||^2 + (\Delta\alpha)' + \mathrm{Ric}(\nabla\alpha, \nabla\alpha) = 0$. If $r(x) =$ distance (x, p) is differentiable at x along the minimizing geodesic joining p to x, it follows that

$$||Hr||^2 + (\Delta r)' + \mathrm{Ric}(\partial_r, \partial_r) = 0. \tag{22}$$

Now, since $Hr(\partial_r, \partial_r) = 0$, the hessian of r has an eigenvalue equal to zero. Then

$$||Hr||^2 \geq \frac{(\Delta r)^2}{n-1}. \tag{23}$$

Substituing Eq. (23) in Eq. (22), we obtain

$$(\Delta r)' + \frac{(\Delta r)^2}{n-1} + \mathrm{Ric}(\partial_r, \partial_r) \leq 0. \tag{24}$$

Using the hypothesis of the theorem, and asserting that $\mathrm{Ric}(\partial_r, \partial_r) \geq k(n-1)$, Eq. (24) gives

$$(\Delta r)' + \frac{(\Delta r)^2}{n-1} + k(n-1) \leq 0. \tag{25}$$

It is immediate to see that the function

$$\psi = (n-1)\sqrt{|k|}\coth(\sqrt{-k}r), \quad k < 0, \tag{26}$$

satisfies the equation $\psi' + \frac{\psi^2}{n-1} + k(n-1) = 0$.

Now, let $\tau : [0, s) \to \mathbb{R}$ be a function that satisfies $\tau(0) = 0$ and $\psi(\tau(t)) = \Delta r(t)$. But since

$$(\Delta r)' + \frac{(\Delta r)^2}{n-1} + k(n-1) \leq 0 = \psi'(\tau(t)) + \frac{\psi^2(\tau(t))}{n-1} + k(n-1), \tag{27}$$

then $(\Delta r)' \leq \psi'(\tau(t))$, which implies

$$\Delta r'(\tau(t))\tau'(t) \leq \psi'(\tau(t)). \tag{28}$$

and $\psi'(t) < 0$. Therefore $\tau'(t) \geq 1$, $\tau(t) \geq t$ and

$$\Delta r(t) = \psi(\tau(t)) \leq \psi(t), \tag{29}$$

since ψ is a decreasing function (Escobar, 2000). If equality holds in Eqs. (17) and (23) gives an equality, and it only happens when Hr has all its eigenvalues equal, except the zero eigenvalue (that comes from $Hr(\partial_r, \partial_r)$). Since $\Delta r = \Delta r_k$, $\forall r \leq s$, Hr has $n-1$ eigenvectors equal to $\psi/(n-1)$.

Supposing that $\{e_i\}$ diagonalize Hr and that $e_n = \partial_r$, from Eq. (6) it follows that

$$D_{e_i}\partial_r = \frac{\psi}{n-1}e_i. \tag{30}$$

At x, $[e_i, \partial_r] = 0$, and then

$$\begin{aligned} K(e_i, \partial_r) &= R(e_i, \partial_r, \partial_r, e_i) = (D_{e_i}D_{\partial_r}\partial_r - D_{\partial_r}D_{e_i}\partial_r - D_{[e_i,\partial_r]}\partial_r)\cdot e_i \\ &= -\frac{(D_{\partial_r}(\psi e_i))}{n-1}\cdot e_i \\ &= -\frac{\psi'}{n-1} - \frac{\psi}{n-1}(D_{e_r}e_i)\cdot e_i \\ &= -\frac{\psi'}{n-1}k - \frac{\psi^2}{(n-1)^2} \\ &= k, \qquad \text{from Eq. (27).} \end{aligned} \tag{31}$$

□

3. SPHERICAL AND HYPERBOLICAL EMBEDDING

In this section we provide a brief exposition of the Robertson–Walker metric and the hyperbolic embedded coordinates.

3.1. The Robertson–Walker Metric

A four-sphere has positive curvature and it satisfies the equation

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_0^2 = R^2, \tag{32}$$

where $\{\xi_A\}_{A=0}^4$ are cartesian coordinates in pseudoeuclidean $\mathbb{E}^{4,1}$ spacetime and R is the four-sphere radius. Then

$$d\xi_0 = -\xi_0^{-1}(\xi_1\, d\xi_1 + \xi_2\, d\xi_2 + \xi_3\, d\xi_3 + \xi_4\, d\xi_4) = -(R^2 - r^2)^{-1/2}(\mathbf{r}\cdot d\mathbf{r}), \tag{33}$$

where $\mathbf{r} = (\xi_1, \xi_2, \xi_3, \xi_4)$ and $r^2 = \mathbf{r}\cdot\mathbf{r}$. On the four-sphere the metric is given by

$$g_s = d\xi_1 \otimes d\xi_1 + d\xi_2 \otimes d\xi_2 + d\xi_3 \otimes d\xi_3 + d\xi_4 \otimes d\xi_4 + d\xi_0 \otimes d\xi_0$$

$$= dr \otimes dr + r^2\, d\Omega \otimes\, d\Omega + \frac{1}{R^2/r^2 - 1} dr \otimes dr$$

$$= \frac{1}{1 - r^2/R^2} dr \otimes dr + r^2\, d\Omega \otimes\, d\Omega,$$

where $d\Omega \otimes\, d\Omega : T_x(S^3) \times T_x(S^3) \to \mathbb{R}$ is the line element on the three-sphere. If we put the time-dependent expansion parameter $a(t)$, the Robertson–Walker metric, written in polar coordinates, is given by

$$g = -dt \otimes\, dt + \frac{a^2(t)}{1 - kr^2} dr \otimes\, dr + a^2(t) r^2 (d\theta \otimes\, d\theta + \sin^2\theta\, d\phi \otimes\, d\phi), \tag{34}$$

where $k = 1/R^2$. If the three-sphere is parametrized by polar coordinates

$$\xi_4 = a \cos\zeta$$

$$\xi_3 = a \sin\zeta\, \cos\theta$$

$$\xi_1 = a \sin\zeta\, \sin\theta\, \sin\phi$$

$$\xi_2 = a \sin\zeta\, \sin\theta \cos\phi, \qquad 0 < \theta, \zeta\, < \pi, \quad 0 \leq \phi < 2\pi,$$

then

$$d\Omega \otimes\, d\Omega \propto\, d\zeta\, \otimes\, d\zeta\, + \sin^2\zeta (d\theta \otimes\, d\theta + \sin^2\theta\, d\phi \otimes\, d\phi). \tag{35}$$

Assuming the map $\sin\zeta\, dr = r\, d\zeta$ (Schrödinger, 1957), it follows the expression

$$\cos\zeta\, = (1 - kr^2)(1 + kr^2)^{-1}. \tag{36}$$

If it is compared with Eq. (34) we see that the metric in Eq. (35) indeed characterizes a spherical manifold.

3.2. The Hyperbolical Embedding

A four-hyperboloid has negative curvature and it can be used to describe de Sitter ($dS_{4,1}$, $dS_{3,2}$) or anti-de Sitter ($AdS_{1,4}$, $AdS_{2,3}$) spacetimes, according to the metric signature. We choose to treat the (4,1)-signature case. It satisfies the equation (Arcidiacono, 2000)

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 - \xi_0^2 = R^2, \tag{37}$$

from which it is immediately seen that

$$d\xi_0 = \xi_0^{-1}(\xi_1\, d\xi_1 + \xi_2\, d\xi_2 + \xi_3\, d\xi_3 + \xi_4\, d\xi_4) = (r^2 - R^2)^{-1/2}(\mathbf{r} \cdot d\mathbf{r}). \tag{38}$$

On the four-hyperboloid surface the metric is given by

$$g_h = d\xi_1 \otimes d\xi_1 + d\xi_2 \otimes d\xi_2 + d\xi_3 \otimes d\xi_3 + d\xi_4 \otimes d\xi_4 - d\xi_0 \otimes d\xi_0$$

$$= dr \otimes dr + r^2 d\Omega \otimes d\Omega + \frac{1}{1 - R^2/r^2} dr \otimes dr$$

$$= \frac{2 - 1/kr^2}{1 - kr^2} dr \otimes dr + r^2 d\Omega \otimes d\Omega,$$

where in this case, $d\Omega \otimes d\Omega$ denotes the metric on the three-hyperboloid.

3.3. The Classical Model

Schrödinger have described the reduced model (Schrödinger, 1957), obtained when one considers the cross section $\xi_3 = \xi_4 = 0$. The $\mathbb{E}^{4,1}$ spacetime becomes a pseudoeuclidean (2+1)-spacetime endowed with Lorentzian metric $g = d\xi_1 \otimes d\xi_1 + d\xi_2 \otimes d\xi_2 - d\xi_0 \otimes d\xi_0$, and the four-hyperboloid given by Eq. (37) becomes

$$\xi_1^2 + \xi_2^2 - \xi_0^2 = R^2. \tag{39}$$

If pseudospherical coordinates are used, in order to parameterize ξ_0, ξ_1, ξ_2, as

$$\begin{aligned} \xi_1 &= R \cos\chi \cosh(t/R) \\ \xi_2 &= R \sin\chi \cosh(t/R) \\ \xi_0 &= R \sinh(t/R), \qquad -\infty < t < \infty, \quad 0 \leq \chi < 2\pi, \end{aligned} \tag{40}$$

a map which is nowhere singular is obtained and it satisfies Eq. (39). The metric is then given by

$$g_r = -R^2 \cosh^2 d\chi \otimes d\chi + R^2 dt \otimes dt. \tag{41}$$

We observe that the new time t varies less rapidly than ξ_0.

In order to recover the full model given by Eq. (39), Schrödinger modified Eq. (40) by two more polar angles, (θ, ϕ) and then the term

$$d\chi \otimes d\chi + \sin^2\chi (d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi) \tag{42}$$

is used to enlarge the metric given implicitly by Eq. (41). Besides, instead of choosing χ as above, if the relation $\sin\chi = \xi_1/R$ is imposed, another map is defined as follows:

$$\begin{aligned} \xi_1 &= R \sin\chi \\ \xi_2 &= R \cos\chi \cosh(t/R) \\ \xi_0 &= R \cos\chi \sinh(t/R). \end{aligned} \tag{43}$$

It reaches another set of pseudopolar angles (χ, t) on the two-hyperboloid. The metric induced by this parametrization is $g_r := -R^2 d\chi \otimes d\chi + R^2 \cos^2\chi dt \otimes dt$. If we want to retrieve the full model again, Eq. (39), we make the metric enlargement given by Eq. (42). This is the so-called static frame of the de Sitter metric. In a more familiar form, the coordinates (ρ, η) are introduced, parametrizing the reduced de Sitter spacetime as

$$\rho = R \sin\chi, \qquad \eta = Rt, \tag{44}$$

which gives

$$g_r = -(1 - \rho^2/R^2)^{-1} d\rho \otimes d\rho + (1 - \rho^2/R^2) d\eta \otimes d\eta. \tag{45}$$

Formally the de Sitter $dS_{4,1}$ spacetime is uniquely given by the $S^3 \times \mathbb{R}$ topology of the four-hyperboloid

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 - \xi_0^2 = R^2 \tag{46}$$

and it can be viewed as a four-sphere $\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_0^2 = R^2$, if a Wick rotation on the ξ_0 coordinate is made: $\xi_0 \mapsto i\xi_0$, where i is the imaginary complex unit. If the inferior hemisphere of the Wick-rotated four-sphere is parametrized, using the coordinates (t, ρ, θ, ϕ) as

$$\begin{aligned}
\xi_0 &= -R(1 - \rho^2/R^2)^{1/2} \cosh(t/R) \\
\xi_1 &= \rho \sin\theta \cos\phi \\
\xi_2 &= \rho \sin\theta \sin\phi \\
\xi_3 &= \rho \cos\theta \\
\xi_4 &= R(1 - \rho^2/R^2)^{1/2} \sinh(t/R),
\end{aligned} \tag{47}$$

where $-\infty < t < \infty,\ 0 < \rho < R,\ 0 < \theta < \pi$ and $0 < \phi < 2\pi$, the (full model) same metric given by Eq. (45) is obtained

$$\begin{aligned}
g = &-(1 - \rho^2/R^2)^{-1} d\rho \otimes d\rho + (1 - \rho^2/R^2)\, dt \otimes dt \\
&- \rho^2 d\theta \otimes d\theta - \rho^2 \sin^2\theta\, d\phi \otimes d\phi.
\end{aligned} \tag{48}$$

3.4. ISOMETRY GROUP GENERATORS AND KILLING VECTORS ASSOCIATED TO $AdS_{4,1}$

Before we obtain a description of the Sitter metric which is equivalent to the Schrödinger model on Minkowski spacetime, we discuss how the $dS_{4,1}$ isometry group emerges from the Killing equations related to this spacetime. We also digress about covariant derivatives and accelerations associated to an arbitrary moving frame on $dS_{4,1}$.

Given the metric (Eq. (48))

$$g = -(1 - \rho^2/R^2)^{-1} d\rho \otimes d\rho + (1 - \rho^2/R^2)\, dt \otimes dt$$

$$- \rho^2 \, d\theta \otimes d\theta - \rho^2 \sin^2\theta \, d\phi \otimes d\phi, \tag{49}$$

let us compute the connection Christoffel symbols of $dS_{4,1}$:

$$\begin{aligned}
&\Gamma^0_{01} = \Gamma^0_{10} = -\Gamma^1_{11} = -\rho R^{-2}(1 - \rho^2/R^2)^{-1} \\
&\Gamma^1_{00} = \rho R^{-2}(1 - \rho^2/R^2), \qquad \Gamma^1_{22} = -\rho(1 - \rho^2/R^2) \\
&\Gamma^1_{33} = -\rho(1 - \rho^2/R^2)\sin\theta, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta \\
&\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \rho^{-1}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta
\end{aligned}$$

where the index notation was changed: $t \mapsto 0$, $\rho \mapsto 1$, $\theta \mapsto 2$ and $\phi \mapsto 3$. Using the above symbols, the acceleration of moving particles is obtained, which follows immediately from the covariant derivatives. First the moving frame $\{\partial_t, \partial_\rho, \partial_\theta, \partial_\phi\}$ and its respective dual frame $\{dt, d\rho, d\theta, d\phi\}$ is chosen in a tangent (cotangent) space at a point on $dS_{4,1}$ spacetime. The following expressions for the covariant derivative D are obtained:

$$\begin{aligned}
D(\partial_\theta) &= -\rho(1 - \rho^2/R^2)d\theta \otimes \partial_\rho + \cot\theta \, d\phi \otimes \partial_\phi + \rho^{-1} d\rho \otimes \partial_\theta \\
D(\partial_t) &= -\rho R^{-2}(1 - \rho^2/R^2)^{-1} d\rho \otimes \partial_t - \rho R^{-2}(1 - \rho^2/R^2)\, dt \otimes \partial_\rho \\
D(\partial_\phi) &= \cot\theta \, d\theta \otimes \partial_\phi - \sin\theta\cos\theta \, d\phi \otimes \partial_\theta - \rho(1 - \rho^2/R^2)\sin^2\theta \, d\phi \otimes \partial_\rho \\
&\quad + \rho^{-1} d\rho \otimes \partial_\phi \\
D(\partial_\rho) &= -\rho R^{-2}(1 - \rho^2/R^2)^{-1}(dt \otimes \partial_t - d\rho \otimes \partial_\rho) + \rho^{-1}(d\theta \otimes \partial_\theta + d\phi \otimes \partial_\phi).
\end{aligned}$$

Applying the covariant derivatives along the orthonormal frame vectors, the respective accelerations are obtained:

$$\begin{aligned}
a_\theta &= D_{\partial_\theta}(\partial_\theta) = -\rho(1 - \rho^2/R^2)\partial_\rho \\
a_t &= D_{\partial_t}(\partial_t) = -\rho R^{-2}(1 - \rho^2/R^2)\partial_\rho \\
a_\rho &= D_{\partial_\rho}(\partial_\rho) = \rho R^{-2}(1 - \rho^2/R^2)^{-1}\partial_\rho \\
a_\phi &= D_{\partial_\phi}(\partial_\phi) = -\sin\theta\cos\theta\,\partial_\theta - \rho(1 - \rho^2/R^2)\sin^2\theta\,\partial_\rho.
\end{aligned}$$

Since $dS_{4,1}$ is a maximally isotropic spacetime, it admits a maximal number, 10, of Killing vectors, which can be obtained immediately from the following Killing equations (see Table I):

$$\begin{aligned}
\partial_\theta \mathbf{u}_\theta + \Omega \mathbf{u}_\rho &= 0 \\
\partial_\rho \mathbf{u}_\theta + \partial_\theta \mathbf{u}_\rho - 2\rho^{-1}\mathbf{u}_\theta &= 0 \\
\partial_\rho \mathbf{u}_\phi + \partial_\phi \mathbf{u}_\rho - 2\rho^{-1}\mathbf{u}_\phi &= 0 \\
\partial_t \mathbf{u}_\phi + \partial_\phi \mathbf{u}_t &= 0 \\
\partial_t \mathbf{u}_\theta + \partial_\theta \mathbf{u}_t &= 0
\end{aligned}$$

Table I. The Killing Vectors Associated to $dS_{4,1}$

p	$\mathbf{u}_{0p}$	$\mathbf{u}_{1p}$	$\mathbf{u}_{2p}$	$\mathbf{u}_{3p}$
1	$\rho\omega \sin\theta \cos\phi\ s$	$-R\omega^{-1} \sin\theta \cos\phi\ c$	$-\rho R\omega \cos\theta \cos\phi\ c$	$\rho R\omega \sin\theta \sin\phi\ c$
2	$-\rho\omega \sin\theta \sin\phi\ s$	$-R\omega^{-1} \sin\theta \sin\phi\ c$	$-\rho R\omega \cos\theta \sin\phi\ c$	$\rho R\omega \sin\theta \cos\phi\ c$
3	$\rho\omega \cos\theta\ c$	$-R\omega^{-1} \cos\theta\ c$	$\rho R\omega \sin\theta\ c$	0
4	$R\omega^2$	0	0	0
5	0	0	0	$-\rho^2 \sin^2\theta$
6	0	0	$\rho^2 \cos\phi$	$-\rho^2 \sin^2\theta \cos\theta \sin\phi$
7	$-\rho\omega \sin\theta \cos\phi\ c$	$R\omega^{-1} \sin\theta \cos\phi\ s$	$\rho R\omega \cos\theta \cos\phi\ s$	$\rho R\omega \sin\theta \sin\phi\ s$
8	0	0	$\rho^2 \sin\phi$	$\rho^2 \sin^2\theta \cos\theta \cos\phi$
9	$-\rho\omega \sin\theta \sin\phi\ c$	$R\omega^{-1} \sin\theta \sin\phi\ s$	$\rho R\omega \cos\theta \sin\phi\ s$	$\rho R\omega \sin\theta \cos\phi\ s$
10	$-\rho\omega \cos\theta\ s$	$R\omega^{-1} \cos\theta\ s$	$-\rho R\omega \sin\theta\ s$	0

Note. We assumed the notation $c = \cosh(t/R)$ and $s = \sinh(t/R)$. The pth line corresponds to the $\mathbf{u}_{qp}$ component of the $dS_{4,1}$ isometry group generators ($p = 1, 2, \ldots, 10$).

$$\partial_t \mathbf{u}_\rho + \partial_\rho \mathbf{u}_t + 2R^{-2}\Omega \mathbf{u}_t = 0$$
$$\partial_\rho \mathbf{u}_\rho - R^{-2}\Omega^{-1}\mathbf{u}_t = 0$$
$$\partial_t \mathbf{u}_t + R^{-2}\Omega \mathbf{u}_\rho = 0$$

where we introduced $\Omega = \rho(1 - \rho^2/R^2)$ and the $\{\mathbf{u}_\mu\}_{\mu=0}^3$ evidently denotes the Killing vectors. It would be desirable to evaluate the Killing equations related to $dS_{4,1}$. From the projective embedding to be presented in Section 5, the isometries of $dS_{4,1}$ can be obtained from the ones of $\mathbb{E}^{4,1}$. The generators of the isometry group of $\mathbb{E}^{4,1}$ are given by

$$\mathbf{u}_A = \Upsilon_A^B \xi_B + \sigma_A, \tag{50}$$

where Υ_A^B and σ_A $(A, B = 0, 1, 2, 3, 4)$ are constants. As already pointed, $dS_{4,1}$ is maximally isotropic and consequently admits a maximal number of Killing vectors, given by (with suitable choices of the Υ_A^B):

$$\mathbf{u}_{AB} = -\xi_A d\xi_B + \xi_B d\xi_A. \tag{51}$$

Using the parametrization given by Eq. (47) we obtain explicitly all the isometry generators shown in Table I.

4. EMBEDDED PROJECTIVE COORDINATES

The metric given by Eq. (45) can be generalized, using embedded projective coordinates. They can be obtained if a stereographic projection of the four-sphere on the Minkowski spacetime is done. It is well known that the Minkowski spacetime can be treated as a tangent space through the four-sphere South pole. This map is the so-called Beltrami (or geodesic) representation (Notte Cuello and Capelas de Oliveira, 1999). To see how to pass from the $(4 + 1)$-dimensional formulation to the (3 + 1)-dimensional orthogonal coordinates (x^0, x^1, x^2, x^3) on the Minkowski spacetime we consider the Beltrami representation, which gives the relation (Gomes, 2002)

$$x^\mu = -R\frac{\xi^\mu}{\xi^4}.$$

Introducing the notation

$$\sigma^2 = x_\mu x^\mu = -x_0^2 + x_1^2 + x_2^2 + x_3^2 \tag{52}$$

and using Eq. (46) we have

$$\xi^4 = -\frac{R}{(1+\sigma^2/R^2)^{1/2}}, \qquad \xi^\mu = \frac{x^\mu}{(1+\sigma^2/R^2)^{1/2}}. \tag{53}$$

By implicit differentiation it follows that

$$d\xi^4 = \frac{x_\mu dx^\mu}{R(1+\sigma^2/R^2)^{3/2}}, \qquad d\xi^\nu = \frac{(R^2+\sigma^2)dx^\nu + (x_\mu dx^\mu)x^\nu}{R^2(1+\sigma^2/R^2)^{3/2}}. \tag{54}$$

Using the above expressions we obtain:

$$g = d\xi_A \otimes d\xi^A = (1+\sigma^2/R^2)^{-1} dx_\mu \otimes dx^\mu \\ + R^{-2}(1+\sigma^2/R^2)^{-2}[x_\mu dx_\mu \otimes x^\mu dx^\mu + 2x^\nu x^\mu dx_\nu \otimes dx_\mu]. \tag{55}$$

This metric is clearly similar to the one given by Eq. (48), if the substitution $\sigma \mapsto i\rho$ is done, obtaining:

$$g = (1-\rho^2/R^2)^{-1} dx_\mu \otimes dx^\mu \\ + R^{-2}(1-\rho^2/R^2)^{-2}[x_\mu dx_\mu \otimes x^\mu dx^\mu + 2x^\nu x^\mu dx_\nu \otimes dx_\mu]. \tag{56}$$

Using Eq. (52) we can write

$$g = (1-\rho^2/R^2)^{-1} d\rho \otimes d\rho \\ + R^{-2}(1-\rho^2/R^2)^{-2}[x_\mu dx_\mu \otimes x^\mu dx^\mu + 2x^\nu x^\mu dx_\nu \otimes dx_\mu]. \tag{57}$$

As can be seen directly, this expression retrieves the Schrödinger description of the static frame of de Sitter metric, Eq. (45), or more generally, Eq. (48), if we consider the metric terms which correspond to restriction on Minkowski spacetime.

5. CONCLUDING REMARKS

We reviewed the LCT in the light of AdS geometry, describing how the metric in AdS spacetime is related to the constant sectional curvature of AdS. Some important features of the topology of AdS are also investigated through the proof of LCT an its previous demonstrated proposition. From the projective splitting of the de Sitter spacetime on Minkowski one we have expressed hyperbolical (de Sitter) coordinates in terms of orthogonal (Minkowski) ones. The choice of the Beltrami representation seems to be the best one adapted to this formulation. Another metric concerning these embedded projective coordinates is constructed, and our formalism retrieves the Schrödinger description of the static frame de Sitter metric, if the terms of the metric restriction on Minkowski spacetime are considered. We have shown that such metric could be obtained from an appropriate parametrization of $dS_{4,1}$, Eq. (47). It would be desirable to obtain the equations of motion, from the Killing vectors explicitly evaluated, but it is the purpose of a forthcoming paper.

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